What Makes Memorization of Basic Arithmetic Facts So Hard? An Investigation of Contributing Cognitive Factors

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An Investigation of Contributing Cognitive Factors

> Let there be no mistake. Arithmetic is still an important part of elementary school mathematics. Children need to know how to add, subtract, multiply, and divide. Children need to learn addition and subtraction facts (such as 8+6=14 and 18-9=9) and multiplication facts (such as 7x6=42)…. As a general rule, parents can expect the following: by the end of second grade, children should know the basic addition and subtraction facts. (NCTM, 2000)

In a world filled with hand-held calculators and computers that can run basic computations in nanoseconds, it may seem surprising that parents, educators and politicians are coming to a consensus that children need to be fluent with basic facts for addition, subtraction, multiplication and division. As technology proliferated in the 1980s, speculation abounded that mathematics education would be able to de-emphasize computation and, in turn, emphasize higher-order aspects of mathematics. Some mathematics educators went so far as to recommend that students should have access to a hand-held calculator at all times for use on classwork, homework, and tests. This extreme position caused a great deal of unease, and certainly contributed to the math wars that occurred in California (and other parts of the country) in the 1990s (Rosen, 2000).

Research findings from the late 1980s provide support for the long-term importance of acquiring basic facts fluency, including: 1) ability to engage in more advanced mathematics due to a reduction in cognitive load (Bjorklund, Muir-Broaddus, & Schneider, 1990; Folds, Footo, Guttentag, & Ornstein, 1990; Geary, 1994; Hopkins & Lawson, 2002; Siegler & Jenkins, 1989); and 2) increased employability, productivity, and wages (Rivera-Batiz, 1992, as cited in Geary & Bow-Thomas, 1996).

As a result of the California math wars, content standards were created that, among other things, establish an expectation that all children will memorize their basic facts. In first grade, for instance, the standard has been established that "[Students] know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory" (California State Board of Education, 1999, Number Sense 2.1).

This first-grade-basic-facts standard sets an expectation that is at least one to two years earlier than traditional U.S. educational guidelines (see, for example, Ilg & Ames, 1951); it is clearly one year earlier than NCTM's recently-published position, quoted above, that students should know their addition and subtraction facts by the end of second grade. It is also expecting that students can achieve memorization of basic addition and subtraction facts, to twenty, one to two years earlier than many children in the U.S. currently seem to achieve this level of memorization (Fuson, 1992; Geary & Bow-Thomas, 1996; Hopkins & Lawson, 2002).

Evidence exists, however, that under certain conditions children can come close to achieving at least the addition portion of this standard in first grade. In a comparison of primary students attending school in China and the United States, Geary and Bow-Thomas (1996) found that by the end of first grade, Chinese students were retrieving (from long-term memory) their answers to basic-addition-to-20 problems 91% of the time.[1] These findings might lead one to conclude that if Chinese first-graders can accomplish memorization of the addition facts, then U.S. students should be able to do the same. After an extensive analysis of their data, however, Geary and Bow-Thomas suggest that "the structure of Asian language and English language number words influences the development of early numerical and arithmetical competencies" in two different ways. First, most Asian language number words are transparent to their base 10 structure (e.g., eleven is named "ten-one" and twenty-three is named "two-ten-three"), which appears to enable children to develop and flexibly use decomposition as a relatively quick and accurate backup strategy (Fuson & Kwon, 1992). Second, number words in Chinese (and some other Asian languages) can be spoken more quickly than English number words, which allows more

[1] By the end of third grade, children in the U.S. sample were retrieving their answers to addition-to-20 problems 56% of the time; 39% of the time they were counting, and 4% of the time they were using decomposition strategies.

numbers to be retained in working memory, which may influence the types of strategies (finger counting, verbal counting, decomposition) that can be effectively used as backup strategies to direct retrieval (Geary & Bow-Thomas, 1996).[2] Geary and Bow-Thomas also found schooling effects (including more instructional time devoted to mathematics instruction in Chinese primary grades than in United States primary grades) that they argue may "give the Chinese children an early advantage in acquiring some arithmetical competencies"; however, they suggest that "these differences should only influence the early stages of skill acquisition and not the ultimate level of skill that can be achieved in arithmetic" (Geary & Bow-Thomas, 1996, p. 2042).

> If standards-driven reform is to live up to its own great expectations, it is critical that states' standards be high but not unreachable, and specific but not directive. Standards should be clear to all readers, whether they be educators, parents, or students. Teachers must demand that their students meet the standards and must provide the instruction and support students need. (Joftus & Berman, 1998, p. 31)

Based on current research findings that younger English-speaking children[3] do not generally use retrieval as a primary strategy for addition basic facts (Geary and Bow-Thomas (1996) found U.S. children at the end of first grade using retrieval only 28% of the time, and counting 68% of the time), three questions appear to be germane to the issues posed by Joftus and Berman, above. First, is this basic-facts-to-20 memory standard reachable by first graders; second, if it is reachable, what reasons might be causing students to not reach it at this time; and third, what instructional strategies and support may be likely to help U.S. students memorize their addition and subtraction basic facts?

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[2] Research supports a position that strategies for solving basic fact problems require differing amounts of cognitive effort and consume different amounts of working memory. These appear to range, from most cognitively demanding to least: counting, min counting (counting on from the larger number), decomposition, finger folding (found primarily in some Asian students), and direct retrieval from long-term memory (Ashcraft, 1990; Fuson, 1992; Geary & Bow-Thomas, 1996; Siegler & Jenkins, 1989).

[3] French-speaking second grade students have also been reported to "make very great use of [counting] even when faced with simple additions (from +1 to +4) …" (Barrouillet and Fayol, 1998, as cited in Roussel, Fayol, & Barrouillet, 2002).

Is a Basic-Facts-to-Twenty Standard Reachable by English-Speaking First Graders?

As mentioned above, Geary and Bow-Thomas (1996) found that by the end of the school year Chinese first-graders were solving a high percentage (91%) of addition basic fact problems using direct retrieval from long-term memory. In this same study, U.S. first-graders used retrieval for slightly more than one quarter of the same addition basic fact problems. Other studies have reported that English-speaking students are still routinely relying on counting and decomposition strategies in second and third grades (Ashcraft, 1990; Fuson, 1992; Hopkins & Lawson, 2002; Siegler & Jenkins, 1989).

Although there is a substantial amount of data available for first-grade strategies related to basic facts addition, less research appears to have been done for basic facts subtraction. In general, the work that has been done establishes that subtraction is more difficult for students to learn than addition, and children use less sophisticated strategies for subtraction in relation to their addition strategies (see Fuson & Fuson, 1992). Although not focused on issues of retrieval, Fuson (Fuson, 1986) found that first-grade students could learn to successfully answer subtraction problems through 18 when taught to use a counting-up procedure, compared to first- and second-grade students who experienced difficulty solving subtraction basic fact problems using counting-down and decomposition strategies (Thornton, 1990, as cited in Fuson, 1992). Studies such as this seem to suggest that researchers are finding little evidence that subtraction facts are being memorized by first and even second graders.

Studies of English-speaking adults provide evidence that many still use a combination of retrieval, counting, and decomposition strategies. Geary and Wiley (1991, as cited in Hopkins and Lawson, 2000) found that college undergraduates used retrieval on average 88% of the time, while elderly adults (averaging 71 years of age) used retrieval on 98% of the same addition basic fact problems. LeFevre et al. (1996, as cited in Hopkins and Lawson, 2000) found that 81% of their sixteen participants reported solving some basic fact problems with strategies other than retrieval. Hecht (1999, as cited in 2002) found that over half of the 61 adult subjects used counting at least once when asked to answer simple addition basic fact problems.

Although the data is relatively sparse, it appears that older adults (such as those reported by Geary and Wiley, 1991, as cited in Hopkins and Lawson, 2000) may be more likely to use direct retrieval to solve basic fact addition problems than younger adults. If this is true, the question arises as to whether it is because of the way they learned their facts to begin with, or whether it is because they have over the course of their lifetimes developed more reliance on retrieval than they might have had when they were younger adults.

Although it is not clear how this question could easily be tested, there is some evidence to suggest that first-grade students in the 1930s and the 1950s were not performing in significantly different ways than the students of today. For instance, Ilg and Ames (1951) report that students at seven years of age (many first-graders are seven by the end of the year, although some won't be seven until they are in second grade) are still counting on some addition problems to twenty. They report that students at eight years of age (2nd and 3rd graders of today) know most addition combinations by heart, although some still use decomposition strategies; they also report that most eight-year-olds know their smaller subtraction combinations by heart, but either "wildly" guess or use decomposition strategies on the "teen" combinations. Finally, Ilg and Ames assert that by the age of nine (3rd and 4th graders of today), students know their addition and subtraction combinations by heart. Even in the 1920s, it appears that addition and subtraction were not expected to be completely learned by the end of first grade; Thorndike (1921) reports practice of addition and subtraction basic facts in textbooks for students all the way through 6th grade.

Based on the evidence discussed above, there is little support for a belief that memorization of addition or subtraction facts to twenty has been achieved by many

American children by the end of first-grade. However, the success of Chinese students as reported by Geary and Bow-Thomas (Geary & Bow-Thomas, 1996) suggests that young children have the cognitive capacity to memorize the addition basic facts if conditions and instructional practices are supportive. Arguments that first-graders lack developmental readiness to understand or memorize basic addition facts seems to be contradicted by the achievement of Chinese students. However, there is not equivalent evidence to support a similar conclusion for subtraction. It seems possible that learning and memorization issues for addition and subtraction are distinctly different. Because it is less clear that subtraction memorization up to twenty is potentially as reachable as addition, the remainder of this paper will focus on issues relating to memorization of addition basic facts.

What Reasons Might Be Contributing to U.S. Students Not Reaching the Addition Standard?

Several different factors may be contributing to low addition basic fact memorization rates of U.S. first grade students, including: 1) working memory demands that interact negatively with slow language word number vocalization; 2) word number teen names that cause confusion in young learners (such as thirteen meaning, but not saying, one ten and three ones) and also don't support decomposition around tens; 3) difficulty encoding basic fact triples into long-term memory; 4) transformations during storage in long-term memory that might potentially cause encoded basic fact triples to reform the triples in ways that may be less useful; and 5) difficulties in retrieving encoded basic facts information. Each of these potential contributing factors will be discussed in turn.

Working Memory Demands

As mentioned above, English number words are relatively lengthy to vocalize, particularly in comparison to Asian number words. As a result, fewer number words can be held in working memory, which may be particularly troublesome for young children who likely have just a few working memory spaces at the age of six and seven. Geary and Bow-Thomas (1996) identified language differences as one significant factor in the differential retrieval rates between Chinese and American students; 10-based decomposition back-up strategies appear to be associated with Asian students who routinely achieve memorization by the end of first grade.

Educational Implications. In a study of two first-grade classrooms in a mid-west private school, Cotter (1996) had students name numbers in ways similar to Asian word number names (e.g., ten-four for fourteen) and count using these ten-based number names for several months. Even though Cotter did not assess students' basic facts acquisition (her study focused on place value acquisition instead), she did find that students were able to transition between their knowledge of English number words and ten-based number words with apparent ease.

Although Cotter's study incorporated several other treatment factors above and beyond having students name numbers with ten-based words, students apparently did develop a more robust understanding of place value. It seems plausible that student use of ten-based number words might facilitate both place-value understanding and increased ability to use ten-based decomposition strategies and facilitate memorization and retrieval of addition basic facts. This hypothesis could be tested using a quasi-experimental design.

Encoding Basic Facts into Long-Term Memory

First-grade teachers will often identify an important learning objective for their students as "students will memorize their basic facts". They select activities, ranging from manipulatives, worksheets, flash cards and timed tests to songs and finger plays, to help students achieve this memorization, which cognitive scientists refer to as "encoding". Several theories have been advanced to explain the encoding process. I will discuss two of these theories, the duplex theory that was the prevailing theory for many years, and the depth of processing theory that is apparently gaining favor.

The Duplex Theory. Based on a theory of two distinct memory structures developed in the 1960s, selected portions of sensory information resided in short-term memory for varying amounts of time (see Anderson, 2000). As information was rehearsed in working memory multiple times, it was believed that the probability that it would be transferred to long-term memory increased. Information could be lost to working memory, and thus never transferred to long-term memory, if it was actively rehearsed (decay), or if additional information was acquired in working memory that exceeded the capacity of working memory (interference). The quantity of rehearsals was believed to be the key to transferring information from short-term to long-term memory. From this perspective, memorizing the basic facts would be almost entirely about rehearsing the facts over and over again.

The Depth of Processing Theory.. An alternative theory, proposed by Craik and Lockhart (1972, as cited in Anderson, 2000), relies on evidence contradicting the concept of two separate memory structures. Instead, this theory argues that the strength of memories ranges on a continuum depending on the depth and meaningfulness of the rehearsal. Thus encoding occurs as a result of rehearsing information in a "deep, meaningful way" (Klatzky, 1980, p. 115-116). This Type II rehearsal, also known as elaborative rehearsal, is different from rote, or maintenance, rehearsal (Type I), where an acoustic representation of the information may be repeated over and over, but no associations between bits of information are developed. Glenberg, Smith and Green (1977, as cited in Klatzky, 1980) conducted a series of studies where subjects repeated words in a rote fashion from 2 to 18 times; these studies found low recall irrespective of the number of repetitions.[4]

The process of "thinking about items, interpreting them, and relating them to other information in long-term memory [mediation] ... prove[s] remarkably effective at enhancing later recall of information from long-term memory" (Klatzky, 1980, p. 119-120). Chunking is one form of mediation that forms one unit of information from several "smaller" units.

Although teachers undoubtedly know quite a bit, at least intuitively, about elaborative rehearsal from their own experiences memorizing materials throughout their educational careers, it is not clear whether they attempt to help their young first-grade

[4] Glenberg, Smith and Green (1977) also found that interacting with the same word on more than one trial resulted in improved recall.

students learn about these Type II rehearsal strategies. However, Paris et al. (1982, as cited in Folds et al., 1990) found that young children, who tend to rely on rote rehearsal, can be taught to use elaborative rehearsal strategies that support improved ability to memorize.

Practice Effects. In the Depth of Processing theory, encoding information is not a question of “memorized” or “not memorized”. Instead, it is a question of how strong an activation level of a memory trace has been developed at any moment in time. “It turns out that the amount of activation spread to a memory depends on the strength of that memory. . . . [and] the more a memory is practiced, the stronger it will become” (Anderson, 2000, p. 186). This practice effect has been identified to be a power function, where the most powerful effect of practice occurs during the initial practice sessions; additional practice sessions continue to contribute to the strength of the memory trace for some time, but these additional practices contribute less and less each time to the strength of the memory trace (see Figure 1).

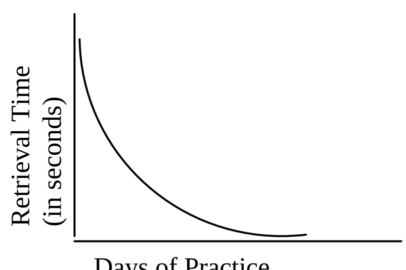

Figure 1. Sample Power Law of Learning Graph

Basic Fact Encoding Models. The experimental support for the duplex and depth of processing theories discussed above appears to be primarily based on memorization tasks involving sets of words, numbers, or nonsense syllables. A number of theories specific to memorization of arithmetic basic facts (addition, subtraction, multiplication, and division) have also been proposed.

The ACT-R Theory, described in Anderson (1983, 1993, as cited in Roussel et al., 2002), focuses on units of knowledge that are encoded and subsequently activated in declarative memory chunks (e.g., 3+5=8 and also 3x5=15). The probability of retrieving one component of a chunk is, in this model, dependent on the strength of activation among the various parts of the chunk. The speed with which a component of a chunk can be retrieved is also dependent on the activation strength.

Of particular interest in the ACT-R theory, and other related theories, is the interference effect that is found among related facts such as 5+8 and 5+9 and is also found between same-operand facts such as 5+8 and 5x8. These interference effects have been identified by comparing retrieval times (RTs) for related facts. When studying problems using the same operation, such as addition, U.S. subjects have routinely evidenced longer RTs for larger combinations (e.g., 7+9) and shorter RTs for smaller combinations such as 2+3 (Ashcraft, 1995; Tronsky, 2001). This problem size effect has been examined in a number of different studies, and appears to be reduced, although not easily eliminated, through extensive practice (Tronsky, 2001).[5] Historically, this problem size effect was attributed to significantly higher proportions of textbook practice problems for smaller problems than for larger problems (see Thorndike, 1921). More recently, however, Lefevre, Sadesky & Bisanz, (1996, as cited in Tronsky, 2001) found that only 5% of the variance in retrieval latencies could be attributed to textbook frequencies.

Several different hypotheses have been advanced to explain the interference effects noted above. One oft-cited theory, originally named the Distribution of Associations theory[6] (Siegler & Jenkins, 1989) suggests that children are more likely to calculate incorrect answers for larger addition and subtraction problems. This is thought to occur because children frequently use counting or decomposition strategies to find their own answers to addition and subtraction facts, whereas they may not use these error-prone strategies with

[5] It should be noted, however, that Chinese third-graders have been found to have a reversed problem-size effect wherein their RTs are shorter for larger problems and longer for smaller problems (Geary & Bow-Thomas, 1996).
[6] This theory was revised by Siegler and Shipley in 1996 to incorporate a system of strategy selection determined by two key factors: individual accuracy with that strategy for a particular problem and length of time required to execute the strategy. Hopkins and Lawson (2002) proposed a further modification of these theories to incorporate counting strategies and the interaction of inefficient and efficient strategies.

larger multiplication problems because of the time and effort involved (Tronsky, 2001). If repeated rehearsal increases the strength of the memory trace (Klatzky, 1980) that then leads to a higher activation strength, each error in computation of a basic fact is proposed to cause increased strength in incorrect chunks of information (e.g., 5+8=14). For larger facts, then, several closely-related “answers” may have relatively similar activation strengths, which can cause slower RTs and larger error rates as well.

A second encoding interference theory suggests that those facts that are learned first (normally the smaller facts) interfere with the ability to memorize facts learned later because the components of the chunks are so similar (Tronsky, 2001). A third interference theory, which may relate more to retrieval issues than to those of encoding, relates to Campbell’s theory of analogue magnitudes (Tronsky, 2001). This theory is based on a “number-line-in-the-head” representation of numbers. Because distances between small numbers are theorized to be easier to distinguish than those same distances between larger numbers, arithmetic problems with smaller solutions should be easier to distinguish from each other than problems with larger answers.

All of the encoding interference theories that have been proposed over the last few decades are now being questioned because we have evidence that at least some Asian children are able to memorize their addition facts with little or no problem size effect and extremely low error rates. In a study comparing college-age Chinese and American students, Geary (as cited in Tronsky, 2001) found that 56-78% of the problem size effect in the American students could be accounted for by lack of practice. Although this is certainly significant, it does not completely explain the variance between Chinese and American students. As mentioned above, base-ten language transparency and number word speed may also account for some of the variance. More importantly, perhaps, Chinese teachers provide their students with different basic fact learning experiences than American teachers tend to offer their students.

One major difference that has been found is that Chinese and Korean students are explicitly taught decomposition strategies in their first few years of school (Fuson & Kwon, 1992). In a study of more than 200 Chinese and American children in grades K through 3, Geary and Bow-Thomas (1996) found that when Chinese students did not retrieve the answer to a basic addition problem from long-term memory, they were much more likely to use a decomposition strategy than American children, who tended to rely much more heavily on counting strategies.

This instruction and repeated use of decomposition strategies may reflect a form of elaborative rehearsal structurally different than the rehearsal children experience when solving problems by counting. It may also provide children with a higher frequency of correct answers that can contribute to a more peaked distribution of associations (Hopkins & Lawson, 2002), which theoretically leads to a higher likelihood of retrieving that answer from long-term memory.

Working with two first-grade classrooms, Cotter (1996) implemented several instructional strategies tied to base-ten understanding, including instruction in decomposition strategies for basic addition fact computation. Although it is not possible to parse out the effects of the various strategies used in her study, it should be noted that the children in the experimental class used ten-based decomposition strategies to a much greater extent than counting strategies; this result was significantly different than that found among children in the control classroom.

Transformation During Storage

Once a child has managed to build a strong memory trace for the correct answer to a basic fact problem, an important teaching question arises: For how long can it be retained? Just as there appears to be a “power law of learning” (whereby memory traces are strengthened more during the initial practice sessions, with additional, but smaller levels of activation strength occurring during successive practice sessions), it also appears that there is a “power law of forgetting” (Anderson, 2000, p. 203-206). When a memory trace is not practiced, activation strength appears to decrease, with later delays in practice correlating with smaller levels of decay than earlier delays after initial learning.

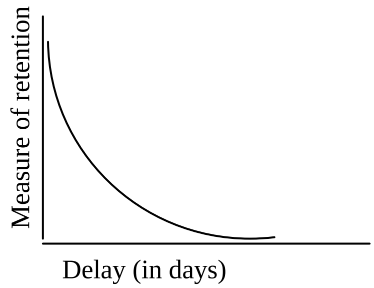

Figure 2. Sample Power Law of Forgetting Graph

This power function for forgetting has important implications in the classroom, particularly in light of the current tendency to teach a new concept or skill for one or two days, and then move on to new material with intermittent review, often several days or weeks between reviews of a particular concept or skill.

Memory Retrieval

In order to retrieve knowledge from long-term memory, it appears that a strong memory trace needs to have been established through repeated elaborative rehearsal, and then maintained with intermittent practice sessions spaced more closely together early in the learning cycle. With these general principles in mind, it also appears that there are additional strategies that can be used to assist in memory retrieval. Two strategies that seem pertinent to basic fact memorization and retrieval have to do with organization and encoding context.

In one word list study, some subjects studied the word lists in organized web-like structures, while other subjects studied the word lists in more random combinations. The retrieval rate for subjects in the organized condition was approximately three times the rate for subjects who learned their word lists in random order (Bower et al., 1969, as cited in Anderson, 2000). Thus it appears that there are definite advantages to learning new material in a hierarchical, or organized, order. Exactly how to structure arithmetic facts to take advantage of this principle is problematic, given the matrix nature of basic facts. Future research could contribute to this potential aid to both encoding and retrieval.

A second intriguing possibility for assisting in retrieval efforts is a concept termed "state-dependent learning." Studies have shown that "[p]eople find it easier to recall information if they can return to the same emotional and physical state they were in when they learned the information" (Anderson, 2000, p. 226-227). Although the studies cited involved subjects learning in different rooms, under water, or while intoxicated, one wonders whether children might be better able to remember their sevens, say, if they learned them while drinking 7-Up. While this is a potential research opportunity; it is certainly not a theory that I would feel ready to suggest to teachers.

What instructional strategies and support may be likely to help U.S. students memorize their basic addition and subtraction facts?

Based on the above analysis, two key issues have surfaced. First, while research supports early memorization of addition basic facts to twenty, there is no comparable evidence that young children in any country are at the same time able to memorize their subtraction basic facts to twenty. Even though math-savvy adults see subtraction as simply addition in reverse, evidence suggests that first-grade students do not easily grasp this concept (Fuson, 1986, 1992). Thus, a first recommendation to mathematics educators and standards policy-makers would be to restrict the target of first-grade memorization to addition facts until and unless research finds evidence that first-graders are actually able to achieve memorization for both sets of basic facts.

Second, instructional strategies commonly presented in U.S. first-grade textbooks and employed in U.S. first-grade classrooms do not appear to conform to the information currently available about memorization. It appears that politicians, textbook authors, parents, and teachers may all harbor similar folk psychologies (Olson & Bruner, 1998) that lead to ineffective folk pedagogies related to helping children learn their basic facts. This may not be particularly surprising, however, because it does not appear that sufficient instructional research has been conducted to help bridge the theory to practice gap. Recent mathematics instruction-based research has focused in large part on instructional strategies focused on helping students develop conceptual understanding and mathematical reasoning skills (see for example Carpenter, Ansell, & Levi, 2001; Fuson, 1986; Fuson & Briars, 1990; Perry, VanderStoep, & Yu, 1993).

Before making wholesale changes to the instructional practices in U.S. first-grade classrooms, it seems wise to examine various classroom interpretations of the theories outlined above. Specifically, such research should examine questions of the following nature:

1) What types of elaborative rehearsal help students build strong memory traces for their basic facts?

2) Should counting strategies by first-grade students be discouraged, because of the potential for error-prone solutions, which appear to flatten association distributions for particular fact combinations?

3) Does explicit instruction in ten-based decomposition strategies help or hinder basic fact memorization?

4) What practice schedules appear to maximize the learning power curve and minimize the forgetting power curve?

5) How many new facts should be introduced in any particular learning session?

6) In what ways can facts be organized to capitalize on possible benefits of hierarchically ordered learning?

Research such as that described above certainly points toward possible answers to questions such as those listed above. In addition to studies into the ways in which memorization of basic facts can be maximized, it also seems important to look carefully at the costs and benefits of emphasizing memorization in the youngest grades. Anecdotally, adults who learned their early arithmetic in other countries share perspectives that address both positive and negative viewpoints relating to the pressures of early memorization.

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