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An Examination of Educational Practices and Assumptions Regarding Algebra Instruction in the United States

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Two potential roadblocks to successful implementation of early algebra in the United States are examined. Systematic algebra instruction has traditionally been reserved for students who have achieved arithmetic proficiency and have shown evidence of having crossed into the formal operational stage as defined by Piaget's stage theory. Current theories regarding these assumptions are discussed. Three developmental models are contrasted with Piaget's theory. Several cognitive obstacles to algebra are analysed in relation to the arithmetic-before algebra expectation. Finally, the paper highlights the importance of dialogue regarding the mental models math educators hold about the appropriateness of early algebra.

Consider two children in 8th grade "Algebra 1," both intelligent and motivated. Ana breezes through the course with confidence and insight; Bonnie struggles with concepts, resorts to memorization of procedures and definitions, and fails all year with the task of developing algebraic thinking and understanding. In the past, U.S. educators would have considered Bonnie to have been placed in Algebra 1 too early–she should have spent another year in "pre-algebra" working on her arithmetic skills and maturing into the formal operational developmental stage (Piaget, 1972) that would allow her to succeed in the abstract, symbolic world of traditional Algebra 1. Today, influenced in large part by international studies such as TIMSS (Schmidt, McKnight, & Raizen, 1997), politicians and educational leaders are requiring all students to begin their study of algebra earlier than ever before. These new requirements for early algebra are in direct conflict with the conventional wisdom of our math educators, who have grown up in a system where the rigorous study of algebra is delayed until students are *ready*, usually in 9th or 10th grade, or in many cases, not at all. Curriculum and instruction decisions at the state, district, and classroom levels are being made based on conflicting philosophies, and these philosophic conflicts must be examined and reconciled before changes that are in the best interests of students can succeed.

Decisions about whether and when to place students in a formal algebra course have centered around teachers', counselors', and administrators' beliefs regarding developmental readiness and arithmetic competency. Educators trained in the Piagetian theory of stage changes have been reluctant to engage students in formal algebraic studies without strong evidence that they have made the stage change from concrete operational to formal operational (Case, 1991a). In addition, most math educators operate under the belief that students must have conquered most of arithmetic's complexities in order to be successful in algebra (Lundin & Bruton, 1999, p. 199; Nathan & Koedinger, 2000a; Nathan & Koedinger, 2000b). Substantial effort has been expended over many years to develop assessments that can distinguish students who have passed both the developmental milestone and the arithmetic hurdle.

Asked what it takes for a student to be successful in algebra, high school teachers often say: "Give me a student who knows his multiplication facts, fraction and integer operations, and who is developmentally ready for abstraction, and I can do the rest." Faced with mounting evidence that students from other countries begin their study of algebra before mastering any of these arithmetic topics (with, perhaps, the exception of the multiplication facts), it is time for our math researchers, curriculum designers, and math teachers to examine both assumptions—that students cannot (or should not) begin their study of algebra 1) until they have entered the formal operational stage defined in Piaget's developmental stage theory, and 2) until they have mastered arithmetic. The following two questions can serve as a frame for this examination and dialogue:

1. Is the acquisition of algebraic concepts and algebraic thinking a consequence of repeated experience over time, or is there a developmental shift of some sort that must occur before understanding can transpire?

2. How do understanding of arithmetic concepts and fluency with arithmetic procedures influence success with algebra?

Algebra Readiness: Developmental Considerations

Piagetian Stage Theory

Many children have difficulty learning algebra, even among the college bound who have traditionally been recommended for the year-long high school Algebra 1 course. One common explanation for the poor algebra achievement rates of 9th grade students is that algebraic understanding requires a stage change of the sort described by Piaget. Concrete operational children, according to Piaget and others, tend to be capable of mental operations as long as they relate to real objects, events, and situations (Wood, 1998). As they mature, they are able to work with more abstract concepts without the aid of concrete objects. Although Piaget (1972) postulated that most children transition from the concrete operational stage to the formal operational stage beginning around age 11 and continuing through successive levels until approximately age 15, there is some evidence that many adults never make this transition to generalization, abstraction, and metacognition (Pintrich, 1990, as cited in Sushkin, 2000). If this theory of stage changes is correct, it is easy to see why many children would struggle with algebra, which many U.S. textbooks and teachers present even to beginners in its most pure form, with few connections to context and concrete objects.

Educators trained in Piaget's theory of cognitive development most often structure their selection of cognitive tasks and curricula based on a belief system that children must be *developmentally ready* before they will be capable of understanding new concepts and mastering new skills. Teachers with this philosophical background have a predisposition to expect that arithmetic is the appropriate precursor to algebra, because arithmetic uses specific numbers that can easily be related to concrete objects and contextual situations. These teachers would expect that the systematic study of algebra at earlier ages would be ineffective because students are not cognitively prepared to handle abstraction. Thus, we can predict that teachers who have developed a Piagetian philosophy of education will be resistant to educational reforms that promote early instruction in algebra.

It is likely that a majority of U.S. teachers believe, consciously or unconsciously, in Piaget's stage theory. A survey of 30 5th and 6th grade California teachers conducted in January, 2001, found that 93% of these teachers remembered studying Piaget's stage theory, whereas only 17% indicated any familiarity with alternative developmental theories. Even for teachers who have not been explicitly trained in Piaget's theories, school structures and textbook

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expectations that implicitly rely on Piaget's theories send a "subliminal" message. Of the 30 surveyed teachers, 73% indicated either high agreement or moderate agreement with the position that "waiting to teach a new topic until children are developmentally ready for that topic will allow them to be more successful.

Observations of teachers in a California school district that implemented a series of algebra inservices for 5th and 6th grade teachers are instructive. Even after more than 25 hours of professional development designed to increase the algebra skills of the surveyed teachers and to provide engaging and age-appropriate lessons for their students, 40% of these teachers agreed that "children need to have entered the stage of abstract thinking before they can successfully master algebra concepts and skills." This belief system was demonstrated several times over the course of the inservices when teachers questioned the appropriateness of spending class time "introducing" algebra when there is clearly not enough time for students to learn all of the state-mandated math standards (which include algebra standards).

Beyond Piaget

Piaget's research and theories have had a profound impact on educational theory and practice in the United States. In addition to his stage theory, Piaget also contributed to the educational philosophy regarding the importance of student construction of knowledge. It is interesting to compare the acceptance (or lack thereof) that each of these two theories has achieved. Although many teachers struggle with the implications of constructivist educational theories, far fewer teachers take issue with his developmental theory. The concept that developmental readiness is determined by maturity seems to fit easily into the historical U.S. structure of universal education, whereby all children were provided a *basic* education in reading, writing and arithmetic, and fewer children were expected to study advanced topics such as algebra.

Unbeknownst to most teachers, however, research findings have demonstrated inconsistencies in Piaget's stage theory. Several competing theories have emerged which either attempt to build on Piaget's basic theory or are based on different assumptions than those of Piaget. Three exceptions to his stage theory demonstrate the reasoning behind current research efforts.

1) Researchers have found a large number of decalages, in which performance tasks predicted by Piaget's theory to be at the same stage of development are passed at very different ages (Case, 1991a). For example, children appear to reach the stage of concrete operations at age 8 or 9 based on a conservation of weight assessment; yet on a conservation of number assessment they appear to reach this stage at age 5 or 6 (Piaget & Inhelder, 1974, as cited in Case, 1991a). This *asynchrony* in cross-task performance has led to a schism in current theories of cognitive development, including the neo-Piagetian, neo-nativist, and information sciences theories discussed below.

2) Research in which children were provided instruction which enabled them to accomplish tasks at a new developmental stage found that this stage change was not universal across all knowledge domains for any particular individual (Case, 1991a). The realization that making a stage change in one knowledge domain does not necessarily transfer to other knowledge domains has caused many theorists to focus on development within specific domains, with little or no reference to "underlying logico-mathematical structures" that Piaget used to explain general stage changes. (Case, 1991b).

3) Cross-cultural studies have also produced data inconsistent with Piaget's stage theory, including evidence that adults in some cultures do not move to the formal operations stage (Dasen, 1972, as cited in Case, 1991a). Moreover,

international mathematics comparisons, including the Third International Mathematics and Science Study (TIMSS) (Schmidt, et al., 1997; Silver, 1998) have shown that children from other cultures are successfully learning algebra concepts and skills at earlier ages than Piaget's stage theory would predict.

As a result of these inconsistencies, current developmental theories have been constructed that: 1) continue to build on the notion of general stage changes, or 2) suggest that developmental stages occur within independent knowledge domains rather than across them, or 3) view development as a continuum. These three different approaches to understanding learning and learning difficulties are exemplified in the following paragraphs.

1) Neo-Piagetians, who build on the developmental theories of Piaget, continue to posit some type of central, "domain-general" structure which exerts limitations on the cognitive tasks children can acquire prior to crossing into a new developmental stage (Carey, 1999; Case, 1991a). Each stage is posited to move an individual to a different way of thinking (Wood, 1998). These limitations are often explained in terms of working memory limitations and processing speed limitations which are thought to improve with age (Pascual-Leone, 1969, as cited in Case, 1991a). Most of the current neo-Piagetian theories, although continuing to include three or four proposed stages of development, also include structures that help explain the differences found in different knowledge domains, based primarily on the context and experiences of individual children.

2) Neo-nativists, building on the theories of Chomsky and Fodor, suggest that the human infant is born with a few modular domains of core knowledge such as knowledge of language, physical objects, persons, and space, already in existence. Those cognitive scientists who argue for the existence of various modular domains, both innate and learned, base their arguments on numerous experiments which show related stage-like changes occurring in the same individual at various ages. For example, a child may show a significant shift in understanding in the area of *naïve* physics months and even years before making the shift to a more mature understanding in the domain of *naïve* biology, or vice versa (Carey, 1988).

Within this neo-nativist school of thought, two theories of cognitive growth have emerged. One possibility is that cognitive development occurs through enrichment, achieved through perceptions of experience and reflection. This view of development, referred to as the "continuity hypothesis," relies on the assumption that the core principles and conceptual framework for any particular individual stay constant over time (Carey, 1999; Keil, 1988). The other possibility is that core principles are abandoned as individuals' experiences and reflections cause "incommensurable differences" between earlier concepts and new knowledge (Carey, 1996). When core principles are abandoned, new ways of thinking and reasoning can occur. These conceptual changes, if they occur, are referred to as "discontinuities."

Evidence for significant, qualitative cognitive shifts in the number domain is found throughout childhood, and may possibly even extend into adulthood. Probably the first candidate for a developmental discontinuity in the number domain is the shift from an infant's innate pre-linguistic representation of number (Carey, 1999; Dehaene, 1997) to an understanding of the count sequence that integrates linguistics with a symbol system. Other potential discontinuities in the number domain include the shift from counting strategies to the ability to decompose and recompose numbers in order to efficiently add and subtract (Fuson, 1992), the difficulty U.S students have understanding place value (Fuson, 1992), and the difficulties many students have enlarging their understanding of numbers to include fractions and decimals (Thornton, 1990, as cited in Fuson, 1992; Greer, 1992, p. 245-246).

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3) Information science theorists, studying differences between novices and experts in any given knowledge domain, have focused attention on the importance of repeated and prolonged experience, as well as the connections between discrete pieces of information that allow domain-specific knowledge to be accessed in larger and larger *chunks.* Individuals with similar memory storage capacities will thus be more expert when their knowledge structures within a given domain are more extensive and more integrated, allowing them to identify underlying structures and hold and utilize more information in working memory because of the chunks of connected knowledge stored as larger units (Case, 1991a).

Rethinking Early Algebra Success Indicators – Experience, Context, and Culture

Today, theories are being proposed that attempt to integrate the most salient parts of the neo-Piagetian, neo-nativist, and novice/expert theories into one coherent model of developmental growth (Case, 1991b). Future research will help determine this next generation of developmental theories. Even before these theories are fully developed and tested, however, it is clear that experience, context, cultural traditions, and language (Wood, 1998, p. 253-256) are critical factors in cognitive development. Although the neo-Piagetians continue to theorize upper limits on learning and task performance due to maturational stages, even they seem to acknowledge the benefit of appropriate instruction in moving a child to the next stage within a particular knowledge domain (Case, Griffin, McKeough, & Okamoto, 1991).

Novice/expert theorists emphasize the critical role experience plays in developing a new area of expertise, and also of the connections that must be made between discrete pieces of knowledge in order to maximize the working memory limitations that exist at all ages, most particularly children under ages 15 (Jensen, 1998, p. 106). This expectation that learners must interact with new knowledge and skills many times, and must make connections between small knowledge units to create larger chunks of useable information and skills, seems to be widespread in the international educational community. However, U.S. math educators have a well-developed belief system that exposing children to algebra prior to age 13 or 14 will be of little or negative benefit because of cognitive limitations imposed prior to formal operations.

Evidence in support of this position appears abundant—relatively few students achieve solid success in traditional 9th grade Algebra 1 classes. However, considering that these students are typically age 14 or even older, Piaget's stage theory does not seem to be the appropriate culprit. More responsibility for the poor success rate in a typical Algebra 1 class may rest on the fact that students are expected to develop algebraic proficiency in one year, spending extremely short periods of time with each new algebraic concept or skill before moving on to new, and often unconnected, concepts and skills. Interestingly enough, we Americans do not expect our children to achieve expertise in music or sports in one short year—instead, we understand that experience and practice must begin early and must be distributed over many years to achieve mastery. The novice/expert theory provides support for the position that the average student needs much more experience with algebra and functions, beginning several years before that mastery is expected. In order for this to happen, mathematics textbook authors and elementary teachers need to examine their tacit assumptions about cognitive development. Piaget's four-stage developmental theory is not supported by data. Teachers, making decisions every day about which mathematical concepts to emphasize and which ones to skip, need current developmental information in order to consider early algebra instruction as a road to success for their students.

A Second Algebra Readiness Question: Arithmetic before Algebra?

Ninth-grade Algebra 1 students not only lack prior experience with algebra; they may also have developed an over-reliance on arithmetic and *means-end* solution strategies. As mentioned above, there is a widespread belief among U.S. educators that arithmetic competence is a prerequisite to successful study of algebra. Since most students continue to struggle with fractions, decimals and integers into 7th and 8th grades, their opportunity to begin serious and sustained algebra experiences is generally postponed until 9th grade or later. Yet evidence from other countries indicates that students can understand significant algebra and functions concepts prior to mastering the complete range of arithmetic skills and concepts (Schmidt, et al., 1997). An analysis of the major obstacles beginning algebra students face may help identify the relative importance of arithmetic skills to their success or failure. If the resolution of these obstacles requires considerable arithmetic mastery, then the U.S. system of arithmetic before algebra may be justified. If, however, these obstacles have little to do with arithmetical proficiency, this information will be useful as educators continue their dialogue about how to improve students' algebraic skills.

Obstacles to Algebraic Success

Educational researchers have catalogued many of the difficulties algebra students encounter, ranging from concatenation issues to problems with graphing intervals (Chaiklin, 1989; Herscovics, 1989; Kieran, 1989). Some of the difficulties researchers have detailed can be resolved fairly easily with additional practice and teacher awareness of the potential misunderstandings. Other difficulties seem to be long-term barriers that individuals find difficult to overcome, even with appropriate instruction and practice. These barriers may qualify, under the neo-nativist modularity theory of differentiated development within individual knowledge domains, as *cognitive obstacles*. Three areas reported to be pervasive obstacles for algebra students are described below:

Algebra Sign Systems

Many beginning algebra students have difficulty expanding their understanding of arithmetic sign systems to include the new sign systems attached to algebra (Kieran, 1989; Kieran, 1992; Wood, 1998, p. 239). One type of sign system difficulty students often experience is a belief that algebraic expressions such as "3x" represent a numeral "3" next to the substitute numeral for "x". For instance, when x equals 5, beginning students will often write "35". Wenger describes students with "extensive algebra experience … [who] cannot seem to "see" the right things in … algebraic expressions and have no clear sense of where they are going" (1987, as cited in Kieran, 1997).

A second significant difficulty related to the symbolic structure of algebra is that of generalizing numerical patterns. Lee and Wheeler (1987, cited in Kieran, 1997) report that less than 10% of 118 second-year algebra students were able to represent a verbal generalization in algebraic notation. When presented with a situation with related unknowns, such as consecutive numbers, students often assign each unknown a different letter (x, y, z), instead of using one letter and expressing the other unknowns in terms of the first (x-1, x, x+1) (Chevallard & Conne, as cited in Kieran, 1997).

A thorough explication of the many difficulties students experience with the algebraic sign system is beyond the scope of this paper. However, difficulties understanding the structure and utility of algebraic symbols do not appear to be dependent on arithmetic computation.

Transitioning from Means-End Strategies to Continuous Translation Strategies

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A second major struggle for students is the need to shift from a means-end problem solving approach to a continuous translation and solution system for problem solving. For instance, many U.S. students will break a word problem into parts; the answer to each part will provide the foundation for solving the next part of the problem.

> "Daniel went to visit his grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother?"

Kieran (1992) reports that sixth graders commonly solve this problem using the equals sign to "announce" the next result: 2.30 + 3.20 = 5.50 – 1.50 = 4.00. An appropriate algebraic translation for this problem would "tell" the story from beginning to end, and would place the unknown at the start of the problem: (x + 1.50) – 3.20 = 2.30. Neither textbooks nor teachers tend to emphasize this more continuous story translation before Algebra 1.

The shift in thinking that students must make in order to move from solving problems in steps (in a means-end strategy of getting one step closer to the solution with each step), to setting up the entire problem first, is dramatic. It may well qualify as a cognitive discontinuity in terms of modularity theory, as it requires an entirely new way of thinking about operations and equations. However, this discontinuity may not be associated with maturation, but may be more related to the current U.S. emphasis on means-end problem-solving in arithmetic word problems. This means-end approach is typically the problem-solving method of choice for U.S. elementary students and teachers alike (Ma, 1999). In fact, many U.S. teachers find it extremely difficult to write algebraic translations of similar word problems, without solving the problem arithmetically and then creating an inverted equation for the problem. Many elementary teachers would instead translate this problem as (2.30 + 3.20) – 1.50 = x. This reliance on *result-unknown* methodologies (Nathan & Koedinger, 2000a) does not appear to be a maturational issue, as researchers have reported that elementary students, with appropriate instruction, are capable of translating similar story problems into appropriate algebraic structures (Cai, 1995, as cited in Cai, 1998; Ma, 1999). Neither does it appear to rely on mastery of higher-level arithmetic skills.

Transitioning from Process to Object Conceptions

A third potential cognitive discontinuity is related to the second—students have great difficulty thinking of algebraic expressions as objects that can be manipulated as if they were individual numbers. Kieran (1992) argues that, "for most people, ... the transition from a 'process' conception to an 'object' conception is accomplished neither quickly nor without great difficulty. Sfard (1991, as cited in Kieran, 1992) suggests that this discontinuity, caused by a "deep ontological gap between operational [procedural] and structural concepts...," is related to maturation. Not only do many adults fail to cross this developmental hurdle, but the historical evidence also supports a centuries-long transition from procedural to structural understanding of algebra (Kieran, 1992). So although this obstacle may not be easily conquered, the problem does not seem to be caused by inadequate arithmetic skills.

Rethinking Algebra Prerequisites

The argument that children must first master arithmetic before they begin a serious study of algebra is not supported by the above analysis. The significant obstacles discussed above relate to sign systems, arithmetic means-end habituation, and the related need to gain procedural confidence prior to transitioning to structural understanding. These obstacles, at least, do not appear to occur because of arithmetic insufficiencies. They seem, instead, to rely on algebraic ways of thinking.

Although high school algebra teachers usually identify rational number operations (primarily integers and fractions) as a critical obstacle to algebraic success, it seems unlikely from the research documentation cited above that these arithmetic skills are the primary impediments to algebraic understanding. In general, teachers working with traditional textbooks do not identify algebraic conceptual issues as causes of student difficulty. For these teachers, algebra is the study of procedures, which students memorize and then imitate, with computation providing the main obstacle to success. When educational researchers, however, examine students' broad understanding of algebra in situations where students do not know which chapter is being assessed, their difficulties with sign systems, algebraic translation, and structural understanding are apparent.

More experience with arithmetic will not help students shift their thinking in these three areas. In fact, as evidenced in international studies (Schmidt, et al., 1997), students who experience algebraic thinking earlier are more likely to make these cognitive leaps. The question arises, then, as to whether so many years spent focusing only on arithmetic may not, in fact, cause an over-reliance on arithmetical understanding of mathematics. In either case, a strong argument can be made for systematic, ongoing algebraic experiences much earlier in a students' educational career.

Conclusion

The call in the United States for more students studying algebra, and more students studying algebra earlier, provides an opportunity and an incentive. We have an opportunity to examine the actions that are currently being taken in response to the mandates for higher levels of algebra achievement. These implementation strategies, and their short-term and long-term results, will provide a wealth of data regarding belief systems, leadership characteristics, and student achievement requirements. We have an incentive to educate teachers, math educational leaders, and math textbook authors and publishers, in the areas of cognitive development. Political and educational leaders have made it clear that they believe intensive teacher training in algebra skills and concepts is the critical element in successfully implementing higher algebra standards. However, to the extent that we disregard the philosophical beliefs of teachers who have grown up in an era of *algebra readiness* expectations, we travel down the road of many other well-intentioned educational reforms. Teachers are the ultimate decision-makers regarding which reform efforts are well implemented. Many of their decisions are based on deep-seated philosophical beliefs, of which even the individual teachers may be unaware.

Researchers and K-12 math leaders continue to propose "algebraic thinking" and "algebraic reasoning" as the appropriate topics for elementary students. Carpenter, Levi, & Farnsworth (2000) express what seems to be the prevailing sentiment:

> This study showed that young students can learn arithmetic in ways that provide a foundation for learning algebra. Recognizing young students' ability to reason algebraically does not suggest that elementary students should learn high school algebra. Rather, this study showed that a broader conception of algebra can be a part of elementary instruction that builds on students' implicit mathematical knowledge and increases their ability to understand, reason, and engage in challenging problem solving. (p.3)

This philosophy speaks to the ubiquitous belief that math educators should provide positive and successful arithmetic and problem-solving experiences for young children until they are *ready* for symbolic algebra. The mandate to provide our students with mathematical opportunities equal to students in other countries is

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in conflict with our philosophical and historical traditions. When curriculum authors and elementary teachers read the new *Principles and Standards for School Mathematics* (NCTM, 2000), an awareness of their underlying beliefs and current research will assist them in interpreting passages such as this with open minds:

> All students should learn algebra. By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school. For example, systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions (p. 37).

Important and far-reaching educational decisions are being made about algebra education in the United States without full awareness of the mental models educators hold about the teaching and learning of algebra. Senge (1990) includes awareness of mental models as one of the five disciplines required of any successful *learning organization*. "Many insights into … outmoded organizational practices fail to get put into practice because they conflict with powerful, tacit mental models" (Senge, 1990, p. 8). If we want children in the United States to have the same mathematical opportunities as children in other parts of the world, math educators of all types (teachers, researchers, curriculum developers, staff developers, and educational policy-makers) need to engage in a dialogue about our mental models surrounding developmental theories and the arithmetic-before-algebra system. With or without this dialogue, changes in the expectations, curricula and teaching of algebra are going to occur. The goal of the dialogue, however, will be to ensure that those changes are thoughtful, informed, and well-implemented in classrooms across the country.

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